N-body Gravity and the Schroedinger Equation

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Abstract

We consider the problem of the motion of N bodies in a self-gravitating system. We point out that this system can be mapped onto the quantum-mechanical problem of an N-body generalization of the problem of the H_2^+ molecular ion in one dimension. We derive a general algorithm for solving this problem, and show how it reduces to known results for the 2-body and 3-body systems.

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1 INTRODUCTION

One of the oldest problems in physics is the N-body problem, which is concerned with describing the motion of a system of N particles interacting through specified forces. It has found applications across a very broad range of fields, including astrophysics, condensed matter physics, plasma physics, nuclear physics and more.

Restricted to the problem of gravitational physics it is particularly challenging, even in the Newtonian case. An exact analytic solution is only known for the $N = 2$ case. For the general relativistic case there is no exact solution in three spatial dimensions even for $N = 2$ (although approximation techniques exist [1]). The main problem here is taking proper account of the dissipation of energy via gravitational radiation, and progress here has relied on an eclectic blend of numerical schemes and approximation techniques.

One of the newest outstanding problems in physics is that of quantum gravity: finding a consistent and predictive theory of relativistic gravitation that is fully quantum mechanical and which reduces in the appropriate (semi-)classical limit to general relativity. The most popular candidates to this end follow the approaches laid out in string theory and loop quantum gravity. However there is currently no fully successful resolution to the problem.

In both cases considerable insight has been attained by reducing the number of spacetime dimensions. Nonrelativistic self-gravitating systems (OGS) of N particles in one spatial dimension have been very important in the fields of astrophysics and cosmology for over 30 years [2], and 2-dimensional quantum gravity has been studied for over 25 years [3]. In the latter case a broad variety of possible theories presents itself. This is because the Einstein-Hilbert action is a topological invariant in two spacetime dimensions, and therefore has trivial field equations. It is therefore necessary to modify two-dimensional relativistic gravity in some way, and the most common procedure for doing so is to incorporate a scalar (dilaton) field. This has the effect of yielding non-trivial equations of motion, albeit at the price of indissoluably coupling the dilaton to gravity and all of the other matter fields in the system. Though this general procedure has antecedants from string theory, it bears a limited resemblance to (3+1)-dimensional general relativity. Furthermore the nonrelativistic ($c \to \infty$) limits of such theories in general bear little resemblance to (1+1)-dimensional non-relativistic gravity [4].

However there is a particular way of incorporating the dilaton that successfully addresses both concerns. The approach involves choosing the coupling so that the field equations couple the stress-energy $T_{\mu v}$of (non-dilatonic) matter to curvature in a manner analogous to that of Einstein gravity in (3+1) dimensions. Since the only measure of curvature in two spacetime dimensions is the Ricci scalar R, it is only the trace of the stress-energy that can act as the source for curvature, suggesting the equation

$$R = \kappa T_{\mu}{}^{\mu} \tag{1}$$

Consequently, as in (3+1) dimensions, the evolution of space-time curvature is governed by the matter distribution, which in turn is governed by the dynamics of space-time [5]. Referred to as $R = T$ theory, it is a particular member of a class of dilation gravity theories on a line (or a circle, pending the choice of topology for an initial data set). Long regarded as a model quantum theory of gravity when $T_\mu{}^\mu$ is constant [3], what makes it particularly interesting is that its non-relativistic limit is that of the Newtonian N-body system when the stress-energy is that of N point particles minimally coupled to gravity. Indeed, it can be regarded as the two-dimensional limit to general relativity [6].

The $R = T$ theory therefore forms an ideal theoretical laboratory not only for quantum gravity, but also for studying both the OGS and its relativistic counterparts. In the latter case there are physical systems with dynamics closely approximated by the one dimensional system. For example very long-lived core-halo configurations are known to exist in the OGS phase space [7]. These are reminiscent of structures observed in globular clusters and model a dense massive core in near-equilibrium that is surrounded by a halo of high kinetic energy stars that interact only weakly with the core. Other higher-dimensional referents include flat parallel domain walls moving in a direction perpendicular to their surfaces and the dynamics of stars in a direction orthogonal to the plane of a highly flattened galaxy. The relativistic OGS (ROGS) has yielded additional insight as to how these properties are modified by general-relativistic effects. The two-body problem has been solved exactly in a variety of physical settings. The 3-body problem has been solved numerically in both the equal [8] and unequal [10] mass cases. Remarkably such a system shows no evidence of increased chaotic behaviour relative to its non-relativistic counterpart, despite the high degree of non-linearity in the system [9] .

The purpose of this paper is to explore the connection between this theory and its potential quantum-mechanical counterpart. We note in particular that the constraint equation of the gravity theory is identical to the stationary Schroedinger equation for a N-body linear atom. We show how to solve these constraint equations to obtain an implicit expression (called the determining equation) for the Hamiltonian in the N-body case, generalizing an approach developed in the 3-body system [8]. In the 2-body case the determining equation can be explicitly solved since it reduces to the defining equation for the Lambert-W function. This yields an expression for the Hamiltonian as a function of the separation of the particles and its conjugate momentum. In the 3-body case the determining equation naturally suggests a generalization of the Lambert W-function, whose properties have been outlined elsewhere [11].

The outline of our paper is as follows. We begin by reviewing the formulation of the N-body problem in (1+1)-dimensional relativistic gravity. The problem reduces to that of solving a single constraint equation. We then discuss how this equation can be mapped onto the Schroedinger equation for an N-body generalization of the H_2^+ molecular ion. We then provide an procedure for solving the constaint equation for an arbitrary number of bodies, and show how it reduces to previously known results for $N = 2$ and $N = 3$. We summarize our work in a concluding section.

2 Basics

The action integral for the gravitational field coupled with N point particles in two spacetime dimensions is [8, 12, 13]

$$\begin{aligned} I &= \int d^2x \left[\frac{1}{2\kappa}\sqrt{-g}\left\{ \Psi R + \frac{1}{2} g^{\mu\nu} \bigtriangledown_\mu \Psi \bigtriangledown_\nu \Psi + \Lambda \right\} \right. \\ & \left. - \sum_{a=1}^{N} m_a \int d\tau_a \left\{ -g_{\mu\nu}(x) \frac{dz_a^\mu}{d\tau_a}\frac{dz_a^\nu}{d\tau_a} \right\}^{\frac{1}{2}} \delta^2(x - z_a(\tau_a)) \right] \end{aligned} \tag{2}$$

where R is the Ricci scalar, $g_{\mu\nu}$ and g are the metric and its determinant, τ_a is the proper time of the a-th particle, $\kappa = 8\pi G/c^4$ is the gravitational coupling, and Ψ is a scalar field called the dilaton. This action describes a generally covariant self-gravitating system (without collision terms, so that the bodies pass through each other), in which the scalar curvature is sourced by the point particles and the cosmological constant Λ.

The field equations are obtained by varying the action with respect to the metric, dilation field, and particle coordinates. After some manipulation this gives

$$R - \Lambda = \kappa T_\mu^{P\mu} \tag{3}$$

$$\frac{d}{d\tau_a}\left\{ \frac{dz_a^\nu}{d\tau_a} \right\} + \Gamma^\nu_{\alpha\beta}(z_a) \frac{dz_a^\alpha}{d\tau_a}\frac{dz_a^\beta}{d\tau_a} = 0 \tag{4}$$

$$\frac{1}{2} \bigtriangledown_\mu \Psi \bigtriangledown_\nu \Psi - g_{\mu\nu}(\frac{1}{4} \bigtriangledown^\lambda \Psi \bigtriangledown_\lambda \Psi - \bigtriangledown^2 \Psi) - \bigtriangledown_\mu \bigtriangledown_\nu \Psi = \kappa T^P_{\mu\nu} + \frac{\Lambda}{2} g_{\mu\nu} \tag{5}$$

where the stress-energy due to the point masses is

$$T^P_{\mu\nu} = \sum_{a=1}^{N} m_a \int d\tau_a \frac{1}{\sqrt{-g}} g_{\mu\sigma} g_{\nu\rho} \frac{dz_a^\sigma}{d\tau_a}\frac{dz_a^\rho}{d\tau_a} \delta^2\left(x - z_a\left(\tau_a\right)\right) \tag{6}$$

and is conserved. Note that (3,4) are a closed system of $N + 1$ equations, which can be solved for the N degrees of freedom of the point masses and the single metric degree of freedom. Consistent with the conservation of $T_{\mu\nu}$, the left-hand side of (6) is divergenceless, yielding only one independent equation to determine the single degree of freedom of the dilaton, whose evolution is thus governed by the evolution of the point masses via (5).

Making use of the decomposition

$$\sqrt{-g}R = -2\partial_0(\sqrt{\gamma}K) + 2\partial_1(\sqrt{\gamma}N^1 K - \gamma^{-1}\partial_1 N_0) \tag{7}$$

where the extrinsic curvature is

$$K = (2N_0\gamma)^{-1}(2\partial_1 N_1 - \gamma^{-1}N_1\partial_1\gamma - \partial_0\gamma) \tag{8}$$

we can rewrite the action in the canonical form

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta\left(x - z_a\left(x^0\right)\right) + \pi\dot{\gamma} + \Pi\dot{\Psi} + N_0 R^0 + N_1 R^1 \right\} \tag{9}$$

where $\gamma = g_{11}$, $N_0 = (-g^{00})^{-\frac{1}{2}}$, $N_1 = g_{10}$, and π and Π are conjugate momenta to γ and Ψ respectively. The quantities N_0 and N_1 are Lagrange multipliers that enforce the constraints $R^0 = 0 = R^1$, where

$$R^0 = -\kappa\sqrt{\gamma}\gamma\pi^2 + 2\kappa\sqrt{\gamma}\pi\Pi + \frac{(\Psi')^2}{4\kappa\sqrt{\gamma}} - \left(\frac{\Psi'}{\kappa\sqrt{\gamma}}\right)' + \frac{\Lambda}{2\kappa}\sqrt{\gamma} - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2}\,\delta\left(x - z_a\left(x^0\right)\right) \tag{10}$$

$$R^1 = \frac{\gamma'}{\gamma}\pi - \frac{1}{\gamma}\Pi\Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma}\delta\left(x - z_a\left(x^0\right)\right) \tag{11}$$

with the symbols ($\dot{}$) and ($'$) denoting ∂_0 and ∂_1, respectively.

The dynamical and gauge degrees of freedom can be identified by writing the generator arising from the variation of the action at the boundaries in terms of $\left(\Psi'/\sqrt{\gamma}\right)'$ and π'. The coordinate conditions can be chosen to be $\gamma = 1$ and $\Pi = 0$; upon elimination of the constraints, the action becomes

$$I = \int d^2x \left\{ \sum_a p_a \dot{z}_a \delta\left(x - z_a\right) - H \right\} \tag{12}$$

where

$$H = \int dx H = -\frac{1}{\kappa}\int dx \Delta\Psi \tag{13}$$

is the reduced Hamiltonian, with $\Delta \equiv \partial^2/\partial x^2$. The field $\Psi = \Psi\left(x, z_a, p_a\right)$ is understood to be determined from the constraint equations which are now

$$\Delta\Psi - \frac{(\Psi')^2}{4} + \kappa^2\pi^2 - \frac{\Lambda}{2} + \kappa\sum_a \sqrt{p_a^2 + m_a^2}\,\delta\left(x - z_a\right) = 0 \tag{14}$$

$$2\Delta\chi + \sum_a p_a \delta\left(x - z_a\right) = 0 \tag{15}$$

where $\pi = \chi'$. The consistency of this canonical reduction can be demonstrated [12] by showing that the canonical equations of motion derived from the reduced Hamiltonian (13) are identical with the canonical field equations.

The procedure for obtaining a solution to the system is as follows. Equations (14,15) first must be solved to obtain Ψ as a function of the phase-space variables (z_a, p_a), subject to the boundary conditions that the Hamiltonian (13) is finite for large $|x|$. Insertion of this result into (13) then yields the Hamiltonian as function of the physical degrees of freedom (z_a, p_a) of the system. Hamilton's equations then can be employed to solve for the evolution of the system. Using these results, the remaining components of the metric can be obtained by solving the field equations that follow from (9) [13].

3 Comparison with the Schroedinger Equation

Eq. (14) is the same as the stationary Schroedinger equation for a single-electron atom with N field sources. Consider first the situation $z_n < z_{n-1} < \ldots < z_1$. This divides the region into $n+1$ regions defined by

$$\begin{array}{lrl} \text{Region} & 0: & z_1 < x \\ \text{Region} & i: & z_{i+1} < x < z_i \\ \text{Region} & n: & x < z_n \end{array}$$

Setting $E_{pi} = \sqrt{p_i^2 + m_i^2}$ and $\Psi = -4 \ln |\phi|$ allows us to rewrite (14) as

$$\bigtriangledown^2 \phi - \frac{1}{4}\left[\kappa^2(\chi')^2 - \frac{\Lambda}{2} + \kappa E_{pi}\delta(x - z_i)\right]\phi(x) = 0 \tag{16}$$

where the summation convention applies to the repeated Latin index.

Eqn. (15) has the general solution

$$\chi = -\frac{1}{4}p_i|x - z_i| - \epsilon X x + \epsilon C_\chi = 0. \tag{17}$$

where X and C_χ are integration constants. The factor of ϵ appears so that behaviour of χ is odd under time reversal; when $t \to -t$, $\epsilon \to -\epsilon$. In each region r, the function χ' is spatially constant

$$\chi' = -\epsilon X - \frac{1}{4}p_i s_{pi,r} \tag{18}$$

where we have defined

$$s_{pi,r} = \begin{cases} 1 & \text{if } i > r \\ -1 & \text{if } i \leq r \end{cases}$$

Insertion of (18) into eqn. (16) yields

$$\bigtriangledown^2 \phi - \frac{1}{4}\left[\kappa^2(\epsilon X + \frac{1}{4}p_i s_{pi,r})^2 - \frac{\Lambda}{2} + \kappa E_{pi}\delta(x - z_i)\right]\phi(x) = 0 \tag{19}$$

or alternatively

$$-\bigtriangledown^2\phi + \frac{1}{4}\left[\kappa E_{pi}\delta(x - z_i)\right]\phi(x) = \mathcal{E}\left(E_{pi}\right)\phi(x) \tag{20}$$

where

$$\mathcal{E}\left(E_{pi}\right) = \frac{\Lambda}{2} - \frac{\kappa^2}{4}(X\left(E_{pi}\right) + \frac{1}{4}\epsilon p_i s_{pi,r})^2 \tag{21}$$

Equation (20) is formally the same as the stationary Schroedinger equation for a particle of mass m and energy $\frac{\hbar^2}{2m}\mathcal{E}$ interacting with a linear molecule whose field sources have charges κE_{pi}. This is an N-body generalization of the problem of the H_2^+ molecular ion in one dimension[14, 15, 16]. We note that there exists a D-dimensional version of H_2^+ , which can be obtained via a scheme called 'dimensional scaling' [17, 18, 19]. The 2-body case of the one-dimensional limit of H_2^+ is solvable in terms of a generalized Lambert W function [13].

4 Solving the Constraint Equations

Eqn. (16) when $x \neq z_a$

$$\bigtriangledown^2 \phi + \frac{1}{4}\phi\left[\kappa^2(\chi')^2 - \frac{\Lambda}{2}\right] = \bigtriangledown^2\phi + \frac{1}{4}\phi\left[\kappa^2(-\epsilon X - \frac{1}{4}p_i s_{pi,r})^2 - \frac{\Lambda}{2}\right] = 0 \quad (22)$$

can be solved in each region, yielding,

$$\phi_r = A_r e^{\frac{1}{2}K_r x} + B_r e^{-\frac{1}{2}K_r x} \quad (23)$$

where

$$K_r = \sqrt{\kappa^2(X + \frac{1}{4}\epsilon p_i s_{pi,r})^2 - \frac{\Lambda}{2}} \quad (24)$$

The solutions (23) can be regarded as 'free particle' solutions. We procede in the same fashion as for a quantum mechanics problem with field sources that are Dirac delta functions. Outside these sources, the particles are 'free' since there is no potential.

This set of equations must satisfy the matching conditions at the locations $x = z_i$. These are

$$\phi_{r-1}(z_r) = \phi_r(z_r) = \phi(z_r) \quad (25)$$

$$\phi'_{r-1}(z_r) - \phi'_r(z_r) = \frac{1}{4}\kappa E_{pr}\phi(z_r) \quad (26)$$

From conditions (25) and (26)

$$A_{r-1}e^{\frac{1}{2}K_{r-1}z_r} + B_{r-1}e^{-\frac{1}{2}K_{r-1}z_r} = A_r e^{\frac{1}{2}K_r z_r} + B_r e^{-\frac{1}{2}K_r z_r} \quad (27)$$

$$A_{r-1}e^{\frac{1}{2}K_{r-1}z_r} - B_{r-1}e^{-\frac{1}{2}K_{r-1}z_r} = \frac{\kappa E_{pr} + 2K_r}{2K_{r-1}}A_r e^{\frac{1}{2}K_r z_r} + \frac{\kappa E_{pr} - 2K_r}{2K_{r-1}}B_r e^{-\frac{1}{2}K_r z_r} \quad (28)$$

Adding eqns (27) and (28) gives

$$A_{r-1}e^{\frac{1}{2}K_{r-1}z_r} = \frac{\kappa E_{pr} + 2(K_r + K_{r-1})}{4K_{r-1}}A_r e^{\frac{1}{2}K_r z_r} + \frac{\kappa E_{pr} - 2(K_r - K_{r-1})}{4K_{r-1}}B_r e^{-\frac{1}{2}K_r z_r} \quad (29)$$

whereas

$$B_{r-1}e^{-\frac{1}{2}K_{r-1}z_r} = -\frac{\kappa E_{pr}+2(K_r-K_{r-1})}{4K_{r-1}}A_r e^{\frac{1}{2}K_r z_r} - \frac{\kappa E_{pr}-2(K_r+K_{r-1})}{4K_{r-1}}B_r e^{-\frac{1}{2}K_r z_r} \tag{30}$$

is obtained upon subtracting them.

The boundary conditions that ensure the Hamiltonian is finite in the regions 0 and n is,

$$\Psi^2 - 4\kappa^2\chi^2 + 2\Lambda x^2 = C_{0/n}x \tag{31}$$

where C_0 and C_n are constants that have to be determined. Eqn. (31) immediately gives

$$A_n = B_0 = 0 \tag{32}$$

Using Eqn. (32) in eqns (29) and (30) with $r = n$ then implies

$$A_{n-1} = \frac{\kappa E_{pn}-2(K_n-K_{n-1})}{4K_{n-1}}B_n e^{-\frac{1}{2}(K_n+K_{n-1})z_n} \equiv \frac{M_{An}}{4K_{n-1}}B_n \tag{33}$$

$$B_{n-1} = -\frac{\kappa E_{pn}-2(K_n+K_{n-1})}{4K_{n-1}}B_n e^{-\frac{1}{2}(K_n-K_{n-1})z_n} \equiv \frac{M_{Bn}}{4K_{n-1}}B_n \tag{34}$$

Defining

$$L_{AAr} = \kappa E_{pr}+2(K_r+K_{r-1}) \qquad L_{ABr} = \kappa E_{pr}-2(K_r-K_{r-1}) \tag{35}$$

$$L_{BAr} = \kappa E_{pr}+2(K_r-K_{r-1}) \qquad L_{BBr} = \kappa E_{pr}-2(K_r+K_{r-1}) \tag{36}$$

$$e_{AAr} = e^{\frac{1}{2}(K_r-K_{r-1})z_r} \qquad e_{ABr} = e^{-\frac{1}{2}(K_r+K_{r-1})z_r} \tag{37}$$

$$e_{BAr} = e^{\frac{1}{2}(K_r+K_{r-1})z_r} \qquad e_{BBr} = e^{-\frac{1}{2}(K_r-K_{r-1})z_r} \tag{38}$$

allows us to write eqns. (29) and (30) as

$$A_{r-1} = \frac{1}{4K_{r-1}}(L_{AAr}A_r e_{AAr} + L_{ABr}B_r e_{ABr}) \tag{39}$$

$$B_{r-1} = \frac{1}{4K_{r-1}}(-L_{BAr}A_r e_{BAr} - L_{BBr}B_r e_{BBr}) \tag{40}$$

For $r = n-1$, eqn.(33, 34) give

$$\begin{aligned} A_{n-2} &= \frac{1}{4^2K_{n-1}K_{n-2}}(L_{AA,n-1}M_{An}e_{AA,n-1} + L_{AB,n-1}M_{Bn}e_{AB,n-1}) \\ &= \frac{M_{A,n-1}}{4^2K_{n-1}K_{n-2}}B_n \end{aligned} \tag{41}$$

$$\begin{aligned} B_{n-2} &= \frac{1}{4^2K_{n-1}K_{n-2}}(-L_{BA,n-1}M_{An}e_{BA,n-1} - L_{BB,n-1}M_{Bn}e_{BB,n-1}) \\ &= \frac{M_{B,n-1}}{4^2K_{n-1}K_{n-2}}B_n \end{aligned} \tag{42}$$

Repeating this for $r = n-2$ yields

$$\begin{aligned}
A_{n-3} &= \frac{1}{4^3 K_{n-1}K_{n-2}K_{n-3}}(L_{AA,n-2}M_{A,n-1}e_{AA,n-2} + L_{AB,n-2}M_{B,n-1}e_{AB,n-2}) \\
&= \frac{M_{A,n-2}}{4^3 K_{n-1}K_{n-2}K_{n-3}}B_n \qquad (43)\\
B_{n-3} &= \frac{1}{4^3 K_{n-1}K_{n-2}K_{n-3}}(-L_{BA,n-2}M_{A,n-1}e_{BA,n-2} - L_{BB,n-2}M_{B,n-1}e_{BB,n-2}) \\
&= \frac{M_{B,n-2}}{4^3 K_{n-1}K_{n-2}K_{n-3}}B_n \qquad (44)
\end{aligned}$$

In this way a pattern clearly emerges, giving the general form for A_{n-r} and B_{n-r} as,

$$\begin{aligned}
A_{n-r} &= \frac{M_{A,n-r+1}}{4^r \prod_{j=1}^{r} K_{n-j}}B_n \qquad (45)\\
B_{n-r} &= \frac{M_{B,n-r+1}}{4^r \prod_{j=1}^{r} K_{n-j}}B_n, \qquad (46)
\end{aligned}$$

where $M_{A,n-r+1}$ and $M_{B,n-r+1}$ are defined by the recurrence relations,

$$\begin{aligned}
M_{A,n-r+1} &= L_{AA,n-r}M_{A,n-r+1}e_{AA,n-r} + L_{AB,n-r}M_{B,n-r+1}e_{AB,n-r} \qquad (47)\\
M_{B,n-r+1} &= -L_{BA,n-r}M_{A,n-r+1}e_{BA,n-r} - L_{BB,n-r}M_{B,n-r+1}e_{BB,n-r} \qquad (48)
\end{aligned}$$

with

$$M_{A,n+1} = 0 \qquad M_{B,n+1} = 1$$

Evaluating the recurrence relations (48) and (47) for the case $r = n+1$, and resorting to eqns. (32) furnishes the relation

$$0 = \frac{M_{B,0}}{4^{n+1} \prod_{j=1}^{n+1} K_{n-j}}B_n \qquad \Longrightarrow \qquad M_{B,0} = 0$$

since B_n cannot be equal to zero.

Furthermore

$$H = -\frac{1}{\kappa}\int dx \, \bigtriangledown^2 \Psi = -\frac{1}{\kappa}\Big[\Psi'\Big]_{\infty}^{\infty} = -\frac{4}{\kappa}\left[\frac{\phi'}{\phi}'\right]_{\infty}^{\infty} = -\frac{4}{\kappa}\frac{1}{2}(-K_0 - K_n)$$

$$H = \frac{2}{\kappa}(K_0 + K_n) = \frac{4}{\kappa}\sqrt{\kappa^2 X^2 - \frac{\Lambda}{2}} \qquad (49)$$

where the last equality holds for the case $p_i|s_{pi,0}| = p_i|s_{pi,n}| = \sum p_i = 0$. This is the centre-of-inertia coordinate system; we can always choose it without loss of generality. Eqn. (49) can always be inverted to give X in terms of the Hamiltonian,

$$X = \pm\frac{1}{\kappa}\sqrt{\frac{\kappa^2 H^2}{4} + \frac{\Lambda}{2}}. \qquad (50)$$

Set $X > 0$ and substituting back into the recurrence relations yields the determining equation for the Hamiltonian.

For $N = 2$ the recurrence relations yield

$$\left(4K_1^2 + [\kappa\sqrt{p_1^2+m_1^2} - 2K_0][\kappa\sqrt{p_2^2+m_2^2} - 2K_2]\right)\tanh\left(\frac{1}{2}K_1|z_1 - z_2|\right)$$
$$= -2K_1\left([\kappa\sqrt{p_1^2+m_1^2} - 2K_0] + [\kappa\sqrt{p_2^2+m_2^2} - 2K_2]\right) \quad (51)$$

which is the determining equation for the Hamiltonian as defined via eq. (50). Note that in this particular case $K_0 = K_2 = \sqrt{\kappa^2X^2 - \frac{\Lambda}{2}}$ in the centre-of inertia system.

The $N = 3$ case is somewhat more tedious, though the procedure is straightforward. The result is

$$\Bigg[\left(\left(\hat{M}_1+\hat{K}_1\right)\left(\hat{M}_3+\hat{K}_4\right)\hat{M}_2 + \left(\hat{M}_1+\hat{K}_1\right)\hat{K}_3^2 + \left(\hat{M}_3+\hat{K}_4\right)\hat{K}_2^2\right)\tanh\left(\frac{\hat{K}_3}{4}z_{32}\right)\tanh\left(\frac{\hat{K}_2}{4}z_{21}\right)$$
$$+\left(\left(\hat{M}_1+\hat{M}_2+\hat{K}_1\right)\left(\hat{M}_3+\hat{K}_4\right)+\hat{K}_3^2\right)\hat{K}_2\tanh\left(\frac{1}{4}\hat{K}_3z_{32}\right)$$
$$+\left(\left(\hat{M}_1+\hat{K}_1\right)\left(\hat{M}_2+\hat{M}_3+\hat{K}_4\right)+\hat{K}_2^2\right)\hat{K}_3\tanh\left(\frac{1}{4}\hat{K}_2z_{21}\right)$$
$$+\left(\hat{M}_1+\hat{M}_2+\hat{M}_3+\hat{K}_1+\hat{K}_4\right)\hat{K}_2\hat{K}_3\Bigg] = 0 \quad (52)$$

where

$$\hat{M}_i = \kappa\sqrt{p_i^2+m_i^2} \quad (53)$$
$$\hat{K}_j = -2\sqrt{\kappa^2\left[X + \frac{\epsilon}{4}\left(\sum_{i=1}^{3}s_{ji}p_i\right)\right]^2 - \frac{\Lambda}{2}}$$

This system for the equal mass case is discussed in detail in ref. [20]. When $\Lambda = 0$ this equation can be solved using a generalization of the Lambert W-function [11].

When the z_i's are not ordered such that $z_n < z_{n-1} < \ldots < z_1$, as can occur when two particles cross one another, we can treat this situation by relabelling the coordinates so that the ordering is in terms of decreasing magnitude. The algorithm just described can then be applied.

When the z_i's are degenerate we are faced with a different situation. In quantum mechanics this corresponds to two charges merging into one. In the case of two bodies (ie. H_2^+) there are two solutions, gerade (symmetric) and ungerade (antisymmetric), whereas for the H atom (the single Dirac delta function) there is only one solution. The gerade solution maps unto the H atom solution continuously for both the energy and the wavefunction. However the ungerade solution goes into the continuum near $R = 1$, where R is the separation between the two nuclei in atomic units (for more details see ref. [[16]]). It can be analytically extended below $R = 1$ but it corresponds to a different branch of the Lambert W-function. This latter energy does not map onto that for the one-dimensional H atom. Furthermore the ungerade symmetry means the wavefunction vanishes at the midpoint; as $R \to 0$ this will not reproduce the proper behaviour for the H atom. In one spatial dimension this problem can be avoided by allowing the particles to cross as discussed in refs. [9] and [10].

5 Discussion

We have shown that the constraint equations of the relativistic one-dimensional self-gravitating N–body problem can be solved exactly, and have described the procedure for carrying this out. We have given explicit results for the $N = 2, 3$ cases. We have also shown that this system is formally equivalent to the stationary Schroedinger equation for a particle of mass m and energy $\frac{\hbar^2}{2m}\mathcal{E}$ interacting with a linear molecule whose field sources have charges κE_{pi}. The interpretation of the terms is different: in the gravitational case the special-relativistic energy κE_{pi} is equivalent to the charge of the field source.

For future work, the obvious problem to consider is quantization. Recent progress on this for the two-body case in the post-Newtonian limit was recently made, and the shift in energy levels caused by quantum-gravitational effects was explicitly computed[22]. In the present case, since we already know the Lagrangian density whose Euler-Lagrange equations are the Schroedinger wave equation, we can obtain a Hamiltonian density and quantize the system. This procedure is akin to '2nd quantization' ; it allows TRANSITIONS between states, with the dilaton itself acting as the agent of transition.

Another avenue to explore is generalization of this work to higher dimensions, ie to the 2+1 case or the 3+1 case. Since dimensional scaling applies to H_2^+ [17, 18, 19], it is conceivable that a scheme of dimensional scaling applying to the gravity problem if dilatons are involved.

Appendix

Here we derive the equations of motion for the N-body system. These can be derived from Hamilton's equaions:

$$\dot{z}_J = \frac{\partial H}{\partial p_J} = \frac{\partial H}{\partial X}\frac{\partial X}{\partial p_J} = \frac{4\kappa X}{\sqrt{\kappa^2 X^2 - \frac{\Lambda}{2}}}\frac{\partial X}{\partial p_J} \tag{54}$$

$$\dot{p}_J = -\frac{\partial H}{\partial z_J} = -\frac{\partial H}{\partial X}\frac{\partial X}{\partial z_J} = -\frac{4\kappa X}{\sqrt{\kappa^2 X^2 - \frac{\Lambda}{2}}}\frac{\partial X}{\partial z_J} \tag{55}$$

The only thing that needs to be done is to determine the partial derivatives of X with repect to p_a and z_a. This can done by differentiating eqn. (50) in conjunction with the recurrence relations.

Before proceeding to the actual derivation, it is worthwhile cleaning up the notation. Differentiating (24) with respect to p_a and z_a gives, $K_r = \sqrt{\kappa^2(-X - \frac{1}{4}\epsilon p_i s_{pi,r})^2 - \frac{\Lambda}{2}}$

$$\begin{aligned}
\frac{\partial K_r}{\partial z_J} &= \frac{\partial X}{\partial z_J}\frac{\kappa^2(X + \frac{1}{4}\epsilon p_i s_{pi,r})}{K_r} = \frac{\partial X}{\partial z_J} D_{K_r} && (56)\\
\frac{\partial K_r}{\partial p_J} &= \left(\frac{\partial X}{\partial p_J} + \frac{1}{4}\epsilon s_{p_J,r}\right)\frac{\kappa^2(X + \frac{1}{4}\epsilon p_i s_{pi,r})}{K_r} = \left(\frac{\partial X}{\partial p_J} + \frac{1}{4}\epsilon s_{p_J,r}\right) D_{K_r} && (57)
\end{aligned}$$

Thus differentating the recurrence relations (48) and (47) with respect to z_j gives

$$\begin{aligned}
\frac{\partial M_{A,n-r+1}}{\partial z_J} = {} & 2\frac{\partial X}{\partial z_J}(D_{K_{n-r}} + D_{K_{n-r-1}})M_{A,n-r+1}e_{AA,n-r} \\
& - 2\frac{\partial X}{\partial z_J}(D_{K_{n-r}} - D_{K_{n-r-1}})M_{B,n-r+1}e_{AB,n-r} \\
& + L_{AA,n-r}\frac{\partial M_{A,n-r+1}}{\partial z_J}e_{AA,n-r} \\
& + L_{AB,n-r}\frac{\partial M_{B,n-r+1}}{\partial z_J}e_{AB,n-r} \\
& + \frac{1}{2}\frac{\partial X}{\partial z_J}(D_{K_{n-r}} - D_{K_{n-r-1}})L_{AA,n-r}M_{A,n-r+1}e_{AA,n-r} \\
& - \frac{1}{2}\frac{\partial X}{\partial z_J}(D_{K_{n-r}} + D_{K_{n-r-1}})L_{AB,n-r}M_{B,n-r+1}e_{AB,n-r}
\end{aligned} \quad (58)$$

$$\begin{aligned}
\frac{\partial M_{B,n-r+1}}{\partial z_J} = {} & -2\frac{\partial X}{\partial z_J}(D_{K_{n-r}} - D_{K_{n-r-1}})M_{A,n-r+1}e_{BA,n-r} \\
& + 2\frac{\partial X}{\partial z_J}(D_{K_{n-r}} + D_{K_{n-r-1}})M_{B,n-r+1}e_{BB,n-r} \\
& - L_{BA,n-r}\frac{\partial M_{A,n-r+1}}{\partial z_J}e_{BA,n-r} \\
& - L_{BB,n-r}\frac{\partial M_{B,n-r+1}}{\partial z_J}e_{BB,n-r} \\
& - \frac{1}{2}\frac{\partial X}{\partial z_J}(D_{K_{n-r}} + D_{K_{n-r-1}})L_{BA,n-r}M_{A,n-r+1}e_{BA,n-r} \\
& + \frac{1}{2}\frac{\partial X}{\partial z_J}(D_{K_{n-r}} - D_{K_{n-r-1}})L_{BB,n-r}M_{B,n-r+1}e_{BB,n-r},
\end{aligned} \quad (59)$$

Repeating for p_J gives

$$
\begin{aligned}
\frac{\partial M_{A,n-r+1}}{\partial p_J} \quad = \quad & 2\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}+D_{K_{n-r-1}})M_{A,n-r+1}e_{AA,n-r} \\
& -2\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}-D_{K_{n-r-1}})M_{B,n-r+1}e_{AB,n-r} \\
& +L_{AA,n-r}\frac{\partial M_{A,n-r+1}}{\partial p_J}e_{AA,n-r} \\
& +L_{AB,n-r}\frac{\partial M_{B,n-r+1}}{\partial p_J}e_{AB,n-r} \\
& +\frac{1}{2}\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}-D_{K_{n-r-1}})L_{AA,n-r}M_{A,n-r+1}e_{AA,n-r} \\
& -\frac{1}{2}\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}+D_{K_{n-r-1}})L_{AB,n-r}M_{B,n-r+1}e_{AB,n-r}
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\frac{\partial M_{B,n-r+1}}{\partial p_J} \quad = \quad & -2\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}-D_{K_{n-r-1}})M_{A,n-r+1}e_{BA,n-r} \\
& +2\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}+D_{K_{n-r-1}})M_{B,n-r+1}e_{BB,n-r} \\
& -L_{BA,n-r}\frac{\partial M_{A,n-r+1}}{\partial p_J}e_{BA,n-r} \\
& -L_{BB,n-r}\frac{\partial M_{B,n-r+1}}{\partial p_J}e_{BB,n-r} \\
& -\frac{1}{2}\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}+D_{K_{n-r-1}})L_{BA,n-r}M_{A,n-r+1}e_{BA,n-r} \\
& +\frac{1}{2}\left(\frac{\partial X}{\partial p_J}+\frac{1}{4}\epsilon s_{p_J,r}\right)(D_{K_{n-r}}-D_{K_{n-r-1}})L_{BB,n-r}M_{B,n-r+1}e_{BB,n-r},
\end{aligned} \tag{61}
$$

Combining these recurrence relations, the derivative of eqn. 50 with respect to z_J and p_J, the equations

$$\frac{\partial M_{B,0}}{\partial z_J}=0 \qquad \text{and} \qquad \frac{\partial M_{B,0}}{\partial p_J}=0, \tag{62}$$

and the boundary conditions,

$$M_{A,n}=M_{B,n}=0 \tag{63}$$

will give a linear equation in $\frac{\partial X}{\partial z_J}$ and another one in $\frac{\partial X}{\partial p_J}$, which can always be solved. Substituting back in eqns. (54) and (55) will give Hamilton's equations in terms of X.

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